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How Reliable are Compositions of Series and Parallel Networks Compared with Hammocks?

V. Drăgoi, S.R. Cowell, V. Beiu, S. Hoară, P. Gaşpar

Vlad Drăgoi*
1. Faculty of Exact Sciences
Aurel Vlaicu University of Arad
Elena Dragoi St., 310330, Arad, Romania
2. Laboratoire LITIS - EA 4108
Université de ROUEN - UFR Sciences et Techniques
Avenue de l'université, 76800 Saint Etienne du Rouvray, France
*Corresponding author: vlad.dragoi@uav.ro

Simon R. Cowell, Valeriu Beiu, Sorin Hoară, Păstorel Gaşpar
Faculty of Exact Sciences
Aurel Vlaicu University of Arad
Elena Dragoi St., 310330, Arad, Romania
{simon.cowell, valeriu.beiu, sorin.hoara, pastorel.gaspar}@uav.ro

Abstract: A classical problem in computer/network reliability is that of identifying simple, regular and repetitive building blocks (motifs) which yield reliability enhancements at the system-level. Over time, this apparently simple problem has been addressed by various increasingly complex methods. The earliest and simplest solutions are series and parallel structures. These were followed by majority voting and related schemes. For the most recent solutions, which are also the most involved (e.g., those based on Harary and circulant graphs), optimal reliability has been proven under particular conditions.
Here, we propose an alternate approach for designing reliable systems as repetitive compositions of the simplest possible structures. More precisely, our two motifs (basic building blocks) are: two devices in series, and two devices in parallel. Therefore, for a given number of devices (which is a power of two) we build all the possible compositions of series and parallel networks of two devices. For all of the resulting two-terminal networks, we compute exactly the reliability polynomials, and then compare them with those of size-equivalent hammock networks.
The results show that compositions of the two simplest motifs are not able to surpass size-equivalent hammock networks in terms of reliability. Still, the algorithm for computing the reliability polynomials of such compositions is linear (extremely efficient), as opposed to the one for the size-equivalent hammock networks, which is exponential. Interestingly, a few of the compositions come extremely close to size-equivalent hammock networks with respect to reliability, while having fewer wires.[a]
Keywords: two-terminal network, series and parallel network, composition, reliability polynomial.

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1 Introduction

One well-known problem in information processing is that of identifying schemes (for a particular given technology) that maximize *reliability*. Reliability is an attribute of a system, a *reliable system* being one which works error-free for extended periods of time—originally an issue only for safety-critical applications. The most common interpretation of network reliability [3], [4], [5] is connectivity-based. In [5], the author emphasizes a variety of interpretations for the reliability of networks, such as all-terminal, k-terminal or two-terminal networks:

> "*It comes as no surprise that hundreds of seemingly natural definitions arise by examining the plethora of different types of networks, causes and types of failures, and levels and types of operation. One should not expect to find a single definition for reliability that accommodates the many real situations of importance.*"
>
> (Charles J. Colbourn)

Obviously, the design-for-reliability problem becomes more challenging as the system grows larger (more complex) and is required to function without interruptions for longer times. Another aspect of interest is that enhancing/maximizing reliability should be done with a limited number of additional (redundant) components. The number of components is the simplest and most obvious *cost function*, but other cost functions (also known as *figures-of-merit*, or FoM) have been proposed and used, such as, e.g., area, power, or energy. It follows that *design-for-reliability* is a constraint optimization problem: maximize system reliability given limited resources (keeping costs as small as possible). This problem permeates way beyond computers into most man-made systems. Nature also seems to rely on reliability principles/schemes at different levels (the most well-known example here being the human brain, having 10^{11} neurons interconnected by 10^{15} axons and dendrites, working over many years).

In the following we shall first of all restrict the scope of our discussions to computers. In this context, reliability was established through five lectures given at Caltech by John von Neumann in January 1952, which were published four years later [22]. The focus was on how to design reliable circuits/computers using unreliable logic gates. The answer was to replicate gates and combine their effects by voting and/or multiplexing. Another take on this topic was advanced four years later by Edward F. Moore and Claude E. Shannon [20], [21]. The major difference was that instead of starting from gates, Moore and Shannon decided to pursue their analysis starting from relays (switching devices). Their results were much more encouraging than [22]. In particular, their device-level scheme:

- can be used with arbitrarily poor devices (i.e., absolutely random switching devices);
- requires redundancy factors which are $10^{(2...3)}$ (2 to 3 orders of magnitude) less than those needed by gate-level schemes.

Clearly, logic gates are made out of switching devices (transistors), hence device-level approaches, such as [20] should be used to enhance the reliability of the gates, before applying gate-level schemes such as those suggested in [22]. Still, this approach was not taken, as over the last few decades, the CMOS transistors have always been reliable enough. With novel nanoscale switching devices and nanoarchitectures under investigation [23], [26] the story is starting to look different [14], [13]. This prospect has triggered our interest in revisiting the work of Moore and Shannon [7]. Their scheme for improving on an unreliable switching device was to replace the device itself by a two-terminal network of identical unreliable switching devices.

That is why we further narrow down the focus of this paper to two-terminal networks. In particular, Moore and Shannon have introduced in [20] and argued in [21] for a particular type of two-terminal networks: *hammock networks*. A thorough comparison of hammock networks with other highly effective specialized networks is called for. For instance, a comparison of hammocks with circulant and Harary graphs (for which optimal reliability has been proven under particular conditions [19], [9], [25]). In any case, regular networks bode well with novel array-based designs including vertical FET, FinFETs [12], gate-all-around FET, and arrays of beyond CMOS devices [6]. Our fresh analyses of small hammock networks [7] (as well as possible extensions [8], [2]) are exact. They have confirmed once again how challenging is to compute the associated reliability polynomials. These suggest that the design-for-reliability process using hammock networks will turn out to be quite involved.

An alternate design option (also mentioned in [20]) advocates for growing larger networks by combining two smaller networks. These can be connected in series, in parallel, or by "composing" them, i.e., replacing each and every element of a network with the other network (translates into composing their associated reliability polynomials). Compositions of hammock networks are mentioned in [20] and [21]. Still, series and parallel networks are easier to evaluate (as their reliability polynomials are simpler [17]), while compositions of series and parallel networks inherit this benefit. It means that one clear advantage of an approach that relies on *composing series and parallel networks* is a simpler design procedure. The fundamental question is how such composed networks perform versus other two-terminal networks of the same size. In particular, in this paper we compare compositions of small series and parallel networks versus hammock networks.

Related work. First of all we mention that this article is an extended version of a conference paper [10]. We bring here new results concerning combinatorial properties of compositions and a more detailed analysis of the various *FoM*s as functions of some specific design requirement, such as number-of-devices and wires. The same technique of composing networks was also used in [1]. There the authors compared hammock networks with compositions of smaller hammock networks. Their results emphasize the merit of composing smaller networks, as they come really close to hammocks while having more efficient algorithms for computing their reliability polynomials.

Our contribution. The main results that we prove in this article can be summarized as follows:

- Compositions of series and parallel networks are planar matchstick minimal networks (see Proposition 4);
- Their *width* w and *length* l, as well as *number-of-devices* n and *wires* ω are related to the Hamming weight of the corresponding binary vector (see Proposition 5 and Theorem 11);
- There is an algorithm that determines whether a matchstick minimal network is a composition of series and parallel. Our solution is a symmetric binary tree decomposition of depth $m = \log_2 n$;
- The reliability polynomial of compositions of series and parallel networks can be computed very efficiently (see Theorem 16).

The article also provides essential simulations, by detailing the reliability polynomials for all compositions of series and parallel networks with $n = 64$ as well as for the two 8-by-8 hammocks. By means of several *FoMs* we give arguments which prove that, in this particular case, hammocks are more reliable than compositions of series and parallel networks. However, the advantage of compositions is undeniable since they come close to hammocks, they have fewer wires for the same number of devices, and more significantly in our opinion, their reliability can be computed efficiently.

Organisation of the paper. The paper is structured as follows. We introduce two-terminal and hammock networks in Section 2. Compositions of series and parallel networks are discussed in Section 3. Section 4 starts by introducing reliability polynomials and several *FoMs* that we are going to use. Afterwards, we determine (exactly) the reliability polynomials for the hammock and composition networks under investigation, and use these for comparative analyses. The paper ends with some conclusions and further directions for research.

2 Two-terminal networks

2.1 Definitions and properties

Definition 1. Let n be a strictly positive integer. We say that $\boldsymbol{N}$ is a two-terminal network of size n, or an n-network if $\boldsymbol{N}$ is a circuit, made of n identical devices, that has two distinguished contacts/terminals: an input or source S, and an output or terminus T.

With any two-terminal network we associate three parameters: *width* w, *length* l, and *size* n. The width w of $\boldsymbol{N}$ is the size of a "minimal cut" separating S from T. The length l of $\boldsymbol{N}$ is the size of a "minimal path" from S to T. The size of a two-terminal network $\boldsymbol{N}$ is related to l and w by:

$$n \geq wl \tag{1}$$

(see Theorem 3 in [20]).

When the equality in eq. (1) holds, we say that $\boldsymbol{N}$ is a minimal network. Even though there are several types of minimal networks, we will study here only matchstick minimal networks $\boldsymbol{N}$. We will denote a matchstick minimal network of width w and length l by $\boldsymbol{N}_{w,l}$. The set of all matchstick minimal networks of size $n = wl$ will be denoted $\mathcal{N}_n$, and we have:

$$\mathcal{N} = \bigcup_{n} \mathcal{N}_n \quad \text{and} \quad \mathcal{N}_n = \bigcup_{w|n} \mathcal{N}_{w,n/w}. \tag{2}$$

Notice that the set $\mathcal{N}_{1,1}$ has cardinality 1, since there is only one two-terminal network with $w = l = 1$, that is the single device network $\boldsymbol{N}_{1,1}$. In the sequel, we will distinguish two subsets of $\mathcal{N}_n$, namely the set of hammocks and the set of compositions of series and parallel networks.

2.2 Hammock networks

Matchstick minimal networks with the well-known "brick-wall" pattern are known as hammocks [20] (see Fig. 1). They can be generated starting from a parallel-of-series PoS network(see Fig. 1) by connecting vertically adjacent pairs of wires by short vertical "matchsticks" (red vertical lines in Fig. 1). If w and l are both even there are two solutions $\boldsymbol{H}_{w,l}$ and $\boldsymbol{H}^{+}_{w,l}$ (see Fig. 1), while otherwise we are left only with $\boldsymbol{H}_{w,l}$ (see [7] for more details).

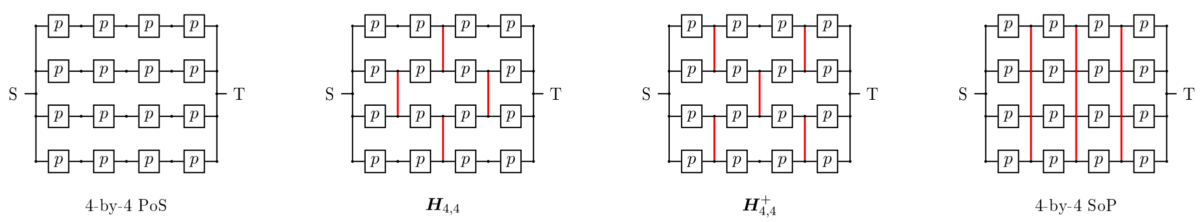


Figure 1: Square 4-by-4 parallel-of-series, hammocks and series-of-parallel.

3 Compositions of series and parallel

3.1 Definitions and properties

Definition 2. Let $\boldsymbol{C}$ represent a composition of networks. If we start from the device itself the simplest possible compositions are: two devices in series $\boldsymbol{C}^{(0)}$, and two devices in parallel $\boldsymbol{C}^{(1)}$. At the to the next level, a composition of $\boldsymbol{C}^{(0)}$ with $\boldsymbol{C}^{(1)}$ is $\boldsymbol{C} = \boldsymbol{C}^{(0)} \bullet \boldsymbol{C}^{(1)}$ which is obtained by replacing each device in $\boldsymbol{C}^{(0)}$ by $\boldsymbol{C}^{(1)}$, with the convention that the nodes S and T in $\boldsymbol{C}^{(1)}$ where unlabeled. This composition will be abbreviated as $\boldsymbol{C}^{\boldsymbol{u}}$, where $\boldsymbol{u} = (0, 1)$.

Notation 3. *More generally, given* $\boldsymbol{u} = (u_0, \ldots, u_{m-1}) \in \{0,1\}^m$*, we will denote by* $\boldsymbol{C}^{\boldsymbol{u}}$ *the composition* $\boldsymbol{C}^{(u_0)} \bullet \cdots \bullet \boldsymbol{C}^{(u_{m-1})}$*, and the set of all such compositions by* $\mathcal{C}_{2^m}$ *(as an example see* $\mathcal{C}_{2^3}$ *in Fig. 2).*

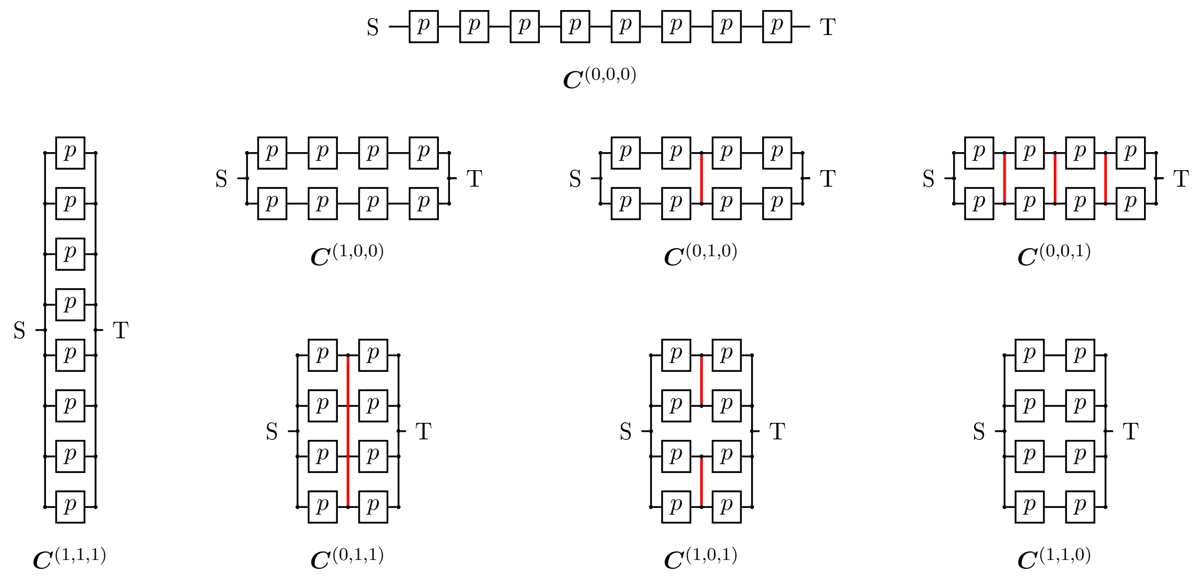


Figure 2: All the elements of the set $\mathcal{C}_{2^3}$.

Theorem 4. *Let m be a strictly positive integer. Then any element in $\mathcal{C}_{2^m}$ is a matchstick minimal network of size 2^m. Moreover, we have $\#\mathcal{C}_{2^m} = 2^m$.*

Proof: The fact that compositions of $\boldsymbol{C}^{(0)}$ and $\boldsymbol{C}^{(1)}$ are matchstick minimal networks follows from Theorem 3 in [20]. The set of compositions of $\boldsymbol{C}^{(0)}$ and $\boldsymbol{C}^{(1)}$ is:

$$\mathcal{C}_{2^m} = \{\boldsymbol{C}^{\boldsymbol{u}} | \boldsymbol{u} \in \{0,1\}^m\}, \tag{3}$$

from which it follows immediately that $\#\mathcal{C}_{2^m} = 2^m$. □

Notice that Proposition 4 provides an efficient method for generating matchstick minimal networks of size $n = 2^m$. Further we will detail how to compute w and l for any network $\boldsymbol{C} \in \mathcal{C}_{2^m}$. To achieve this goal, we introduce the well-known concept of Hamming weight from coding theory. For any binary vector $\boldsymbol{u} \in \{0,1\}^m$, its Hamming weight $|\boldsymbol{u}|$ is the number of non-zero components of $\boldsymbol{u}$.

Theorem 5. *Let m be a strictly positive integer and $\boldsymbol{C}^{\boldsymbol{u}} \in \mathcal{C}_{2^m}$. Then $\boldsymbol{C}^{\boldsymbol{u}}$ is a matchstick minimal network of size 2^m, length $l = 2^{m-|\boldsymbol{u}|}$ and width $w = 2^{|\boldsymbol{u}|}$.*

Proof: For a binary m-tuple $\boldsymbol{u} = (u_0, \ldots, u_{m-1}) \in \{0,1\}^m$ the weight $|\boldsymbol{u}|$ gives the number of times $\boldsymbol{C}^{(1)}$ is present in the composition, i.e., the number of times we compose in parallel. By induction it can be deduced that $w = 2^{|\boldsymbol{u}|}$. Since $\boldsymbol{C}^{\boldsymbol{u}}$ has $n = 2^m$ devices, it follows that $l = 2^{m-|\boldsymbol{u}|}$. □

In conclusion, writing

$$\mathcal{C}_{2^i,2^{m-i}} = \left\{\mathcal{C}_{2^{|\boldsymbol{u}|},2^{m-|\boldsymbol{u}|}} | \boldsymbol{u} \in \{0,1\}^m, |\boldsymbol{u}| = i\right\} \tag{4}$$

we have

$$\mathcal{C}_{2^m} = \bigcup_{i=0}^{m} \mathcal{C}_{2^i,2^{m-i}} \tag{5}$$

3.2 Combinatorial properties

Since compositions of $\boldsymbol{C}^{(0)}$ and $\boldsymbol{C}^{(1)}$ offer an efficient way of creating matchstick minimal networks, the first question that we raise here is to determine the proportion of compositions. In other words, if one randomly picks a matchstick minimal network $\boldsymbol{N} \in \mathcal{N}_{2^m}$,with respect to the uniform distribution over the set of all matchstick minimal networks, then "What is the probability that $\boldsymbol{N} \in \mathcal{C}_{2^m}$?"

Theorem 6. *Let $\boldsymbol{N}$ be a matchstick minimal network of size 2^m. Then we have*

$$Pr(\boldsymbol{N} \in \mathcal{C}_{2^m}) \sim 2^{-(2^{\lfloor m/2 \rfloor}-1)(2^{\lceil m/2 \rceil}-1)+m}.$$

Proof: From [7] we have that for a fixed l and w such that $n = wl$ the number of matchstick minimal networks of length l and width w equals $2^{(l-1)(w-1)}$. Hence the total number of matchstick minimal networks of size 2^m is equal to

$$\sum_{i=0}^{m} 2^{(2^i-1)(2^{m-i}-1)} \sim 2^{(2^{\lfloor m/2 \rfloor}-1)(2^{\lceil m/2 \rceil}-1)}, \tag{6}$$

which ends our proof. □

So, if we randomly pick a matchstick minimal network $\boldsymbol{N}$, the probability that $\boldsymbol{N}$ is a composition of $\boldsymbol{C}^{(0)}$ and $\boldsymbol{C}^{(1)}$ is rapidly decreasing while m is increasing. However, the question now is how to determine whether $\boldsymbol{N}$ is an element of $\mathcal{C}_{2^m}$. To answer this question we define the following two operations

Definition 7 (Vertical/horizontal cut)**.** Let n, w, l be strictly positive integers and $\boldsymbol{N}$ be a matchstick minimal network of width w and length l. We say that $\boldsymbol{N}$ admits a *vertical cut* if there exists a vertical complete matchstick, and we write $\boldsymbol{N} = (\boldsymbol{N}_l|\boldsymbol{N}_r)$. We say that $\boldsymbol{N}$ admits a *horizontal cut* if there is a horizontal free band, i.e. with no matchsticks,and we write $\boldsymbol{N} = \left(\dfrac{\boldsymbol{N}_u}{\boldsymbol{N}_d}\right)$.

Lemma 8. *Let w and l be strictly positive integers and $\boldsymbol{N}$ be a wl matchstick minimal network. Then $\boldsymbol{N}$ can not admit both a horizontal and a vertical cut.*

Proof: The result is straightforward from the definition of a horizontal and vertical cuts (Definition 7). □

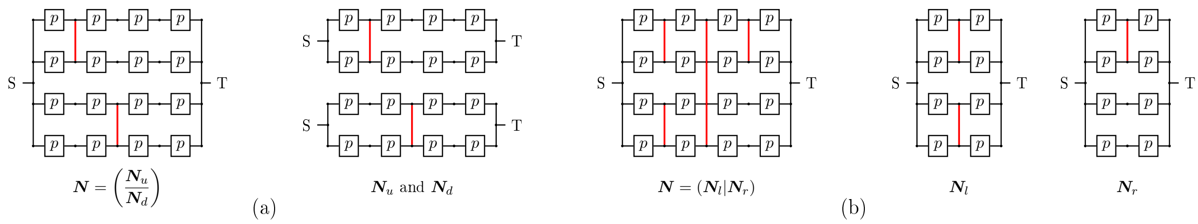


Figure 3: Matchstick minimal networks that admit: (a) horizontal cut; (b) vertical cut.

In this article we only consider those vertical and horizontal cuts that bisect the network into two identical halves. In other words a wl -network $\boldsymbol{N}$ can be cut vertically or horizontally and we write $\boldsymbol{N} = (\boldsymbol{N}_l|\boldsymbol{N}_r)$ with $\boldsymbol{N}_r = \boldsymbol{N}_l$, or $\boldsymbol{N} = \begin{pmatrix} \boldsymbol{N}_u \\ \boldsymbol{N}_d \end{pmatrix}$ with $\boldsymbol{N}_u = \boldsymbol{N}_d$, where $\boldsymbol{N}_l$ is a $w \times l/2$ network and $\boldsymbol{N}_u$ is a $w/2 \times l$ network.

Theorem 9 (Decomposable networks)**.** *Let m be a strictly positive integer and $\boldsymbol{N}$ be a matchstick minimal network of size $n = 2^m$.*

- *if $\boldsymbol{N} = (\boldsymbol{N}_l|\boldsymbol{N}_l)$ (i.e., admits a vertical cut in half), then $\boldsymbol{N} = \boldsymbol{C}^{(0)} \bullet \boldsymbol{N}_l$;*
- *if $\boldsymbol{N} = \begin{pmatrix} \boldsymbol{N}_u \\ \boldsymbol{N}_u \end{pmatrix}$ (i.e., admits a horizontal cut in half), then $\boldsymbol{N} = \boldsymbol{C}^{(1)} \bullet \boldsymbol{N}_u$.*

Moreover, $\boldsymbol{N} \in \mathcal{C}_{2^m}$ if and only if $\boldsymbol{N}$ admits a binary tree decomposition, with respect to "$|$" and "$-$", of depth m, where the leaves of the tree are $\boldsymbol{N}_{1,1}$.

The proof of this proposition is based on the previous Lemma.

Remark 10. Notice that $\boldsymbol{H}_{w,l}$ is not decomposable as a compositions of $\boldsymbol{C}^{(0)}$ and $\boldsymbol{C}^{(1)}$ unless $w = 1$, $l = 1$, or $w = l = 2$, as it does not admit either a vertical or horizontal cut.

Algorithm 1 Decomposition of matchstick minimal networks into composition of $\boldsymbol{C}^{(0)}$ and $\boldsymbol{C}^{(1)}$

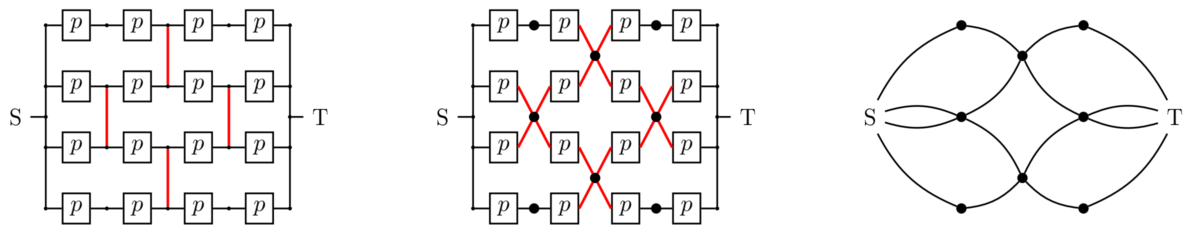
Input: A matchstick minimal network $\boldsymbol{N}$ of size $w \times l = 2^m$
Output: The corresponding binary vector $\boldsymbol{u}$ if $\boldsymbol{N}$ is decomposable

$\boldsymbol{u} = [\quad]$
while $\boldsymbol{N} \neq \boldsymbol{N}_{1,1}$ **do**
    if $\boldsymbol{N} = (\boldsymbol{N}_l|\boldsymbol{N}_l)$ **then**
        $\boldsymbol{N} = \boldsymbol{N}_l$
        Append 0 to $\boldsymbol{u}$
    else if $\boldsymbol{N} = \begin{pmatrix} \boldsymbol{N}_u \\ \boldsymbol{N}_u \end{pmatrix}$ **then**
        $\boldsymbol{N} = \boldsymbol{N}_u$
        Append 1 to $\boldsymbol{u}$
    else
        Break;
    end if
end while

3.3 Representations

In order to compare compositions with Hammocks we consider the parameters of the circuits, that is the number of devices n, as well as the number of wires, ω in the circuits . The first representation of the brick-wall pattern, which is from Moore and Shannon [20], uses vertical matchsticks as in Fig. 1. The second possibility also suggested by Moore and Shannon [20] is to use the graph representation. Here, we will adopt the third representation from [21], which gave the name to these networks: *hammocks*. These three representations can all be seen in Fig. 4.

When counting the number of wires ω we will consider that there are w wires that connect S to the circuit, and w wires that connect T to the circuit. We also count 4 wires whenever we have an "X" shape matchstick. With this convention at hand we can now count ω for a matchstick minimal network, in particular a composition or a hammock.



Brick-wall representation ([20]) Hammock representation ([21]) Graph representation ([20])

Figure 4: Three different representations of $\boldsymbol{H}_{4,4}$.

Theorem 11. *Let m be a strictly positive integer and $\boldsymbol{C}^{\boldsymbol{u}} \in \mathcal{C}_{2^m}$. Then the number of wires of the circuit $\boldsymbol{C}^{\boldsymbol{u}}$ is $\omega = 2^m + 2^{i+1}$, where i is the position of the most significant bit of the corresponding binary vector $\boldsymbol{u}$ equal to 1. When $\boldsymbol{u}$ is the zero vector the number of wires is $\omega = 2^m + 1$.*

Proof: The first case $\boldsymbol{u} = (0, \ldots, 0) \in \{0,1\}^m$ can be easily deduced from the definition of the composition. Next consider $\boldsymbol{u} = (1, 0, 0, \ldots, 0) \in \{0,1\}^m$. This circuit $\boldsymbol{C}^{\boldsymbol{u}}$ is a parallel of two identical circuits $\boldsymbol{C}^{\boldsymbol{v}}$ where $\boldsymbol{v} = (0, \ldots, 0) \in \{0,1\}^{m-1}$. But we know that the number of wires for $\boldsymbol{C}^{\boldsymbol{v}}$ is $\omega = 2^{m-1}+1$ and the number of devices for $\boldsymbol{C}^{\boldsymbol{v}}$ equals 2^{m-1}. We also deduce that there are $2^m/2^{m-1} = 2$ identical blocks in the composition of $\boldsymbol{C}^{\boldsymbol{u}}$. Notice that these two blocks do not share any wire in common. Hence, we obtain the total number of wires for $\boldsymbol{C}^{\boldsymbol{u}}$, that equals the number of blocks times the number of wires in each block. More exactly the number of wires for $\boldsymbol{C}^{\boldsymbol{u}}$ is $\omega = 2 \times \left(2^{m-1} + 1\right) = 2^m + 2$.

Now we can prove our theorem for any $\boldsymbol{u} \in \{0,1\}^m$. Let $\boldsymbol{u} = (u_0, \ldots, u_{m-1})$ be a binary vector such that $u_i = 1$ and $u_j = 0$ for any $j > i$. Denote $\boldsymbol{u}_{i,m-1} = (u_i, \ldots, u_{m-1})$, which equals $\boldsymbol{u}_{i,m-1} = (1, 0, \ldots, 0)$. Notice that $\boldsymbol{C}^{\boldsymbol{u}_{i,m-1}}$ is composed of 2^{m-i} devices and $2^{m-i}+2$ wires. We also know that there are $2^m/2^{m-i}$ identical blocks, all equal to $\boldsymbol{C}^{\boldsymbol{u}_{i,m-1}}$, that do not share any wire in common such that $\boldsymbol{C}^{\boldsymbol{u}}$ is the composition of these blocks. Then the number of wires of $\boldsymbol{C}^{\boldsymbol{u}}$ is $\omega = 2^i \times \left(2^{m-i} + 2\right) = 2^m + 2^{i+1}$.

□

Theorem 12. *Let w and l be two strictly positive integers. Then*

$$\omega = \begin{cases} 2wl - l & \textit{for } \boldsymbol{H}_{w,l} \textit{ with } w \textit{ and } l \textit{ even} \\ 2wl - l + 1 & \textit{for } \boldsymbol{H}_{w,l} \textit{ with } w \textit{ or } l \textit{ odd} \\ 2wl - l + 2 & \textit{for } \boldsymbol{H}^{+}_{w,l} \textit{ with } w \textit{ and } l \textit{ even} \end{cases} \tag{7}$$

Proof: We give here only the proof for one of the cases. For the remaining two cases the arguments are exactly the same. So, let l and w be two odd strictly positive integers. This implies that we have $l-1$ columns of "X" shape matchsticks and horizontal wires. On each one of these columns we count $(w-1)/2$ matchsticks and one horizontal wire. Hence, each column has $4\times((w-1)/2)+1$ wires. And there are $l-1$ such columns, which makes the total number of "interior" wires equal to $(l-1)\times(2w-1)$. By "interior" wire we mean wires that connect only devices and not S or T with any of the devices. Finally, we have to add the number of "exterior" wires, namely those connecting to S and T, which are $2w$. Thus, we obtain $\omega = 2wl - l + 1$. □

Corollary 13. *For square hammocks we obtain*

$$\omega = \begin{cases} 8k^2 - 2k & for \quad \boldsymbol{H}_{2k,2k} \\ 8k^2 - 2k + 2 & for \quad \boldsymbol{H}^{+}_{2k,2k} \\ 8k^2 + 6k + 2 & for \quad \boldsymbol{H}_{2k+1,2k+1} \end{cases} \tag{8}$$

Remark 14. From eq. (8) taking $k = 2^{m/2}$ it follows that $\boldsymbol{H}_{2^{m/2},2^{m/2}}$ has $8\times\left(2^{m/2-1}\right)^2 - 2^{m/2} = 2^m + (2^m - 2^{m/2})$ wires. Also notice that from Theorem 11 there are $\binom{m}{m/2}/2$ elements in $\mathcal{C}_{2^{m/2},2^{m/2}}$ having at most $2^m + 2^{m-1}$ wires, which is smaller than $2^m + (2^m - 2^{m/2})$.

4 Evaluating reliability

4.1 Reliability polynomials

We will use a classical convention for the reliability polynomial $R(p)$, where $p \in [0,1]$ is the probability that a device is closed. Since the polynomial is associated with a network $\boldsymbol{N}$ (either $\boldsymbol{H}$ or $\boldsymbol{C}$), we shall use the notation $R(\boldsymbol{N};p)$. This gives $R(\boldsymbol{C};p)$ and $R(\boldsymbol{H};p)$ for compositions of $\boldsymbol{C}^{(0)}$ and $\boldsymbol{C}^{(1)}$ and respectively hammocks.

Lemma 15. $R(\boldsymbol{C}^{(0)};p) = p^2$ *and* $R(\boldsymbol{C}^{(1)};p) = 1-(1-p)^2$.

This is well-known [22], [20]. We can now determine the reliability $R(\boldsymbol{C};p)$ for any $\boldsymbol{C}$.

Theorem 16. *Let m be a strictly positive integer and $\boldsymbol{u} = (u_0,\ldots,u_{m-1}) \in \{0,1\}^m$. Then:*

$$R(\boldsymbol{C}^{\boldsymbol{u}};p) = R(\boldsymbol{C}^{(u_0)}) \circ \cdots \circ R(\boldsymbol{C}^{(u_{m-1})};p), \tag{9}$$

where $R(\boldsymbol{C}^{(0)};p)$ and $R(\boldsymbol{C}^{(1)};p)$ are given by Lemma 15.

The proof of Theorem 16 follows directly from Definition 2 and Lemma 15.

Remark 17. Notice that compositions of $\boldsymbol{C}^{(0)}$ and $\boldsymbol{C}^{(1)}$ are by definition series and parallel networks. Hence, they inherit all the nice properties of this big family of networks. Series and parallel networks were extensively studied [24], [18], [11] and efficient algorithms exist for computing their reliability polynomials (the complexity of these algorithms is linear in n). However, as we have shown in Theorem 16, compositions of $\boldsymbol{C}^{(0)}$ and $\boldsymbol{C}^{(1)}$ admit a closed form formula of complexity $\log_2(n)$ for computing their reliability polynomials.

Table 1: Reliability polynomials for $\boldsymbol{C^u}$.

$\boldsymbol{N}$	$R(\boldsymbol{N};p)$
$\boldsymbol{C}^{(1,1,1,0,0,0)}$	$8p^8 - 28p^{16} + 56p^{24} - 70p^{32} + 56p^{40} - 28p^{48} + 8p^{56} - p^{64}$
$\boldsymbol{C}^{(1,1,0,1,0,0)}$	$16p^8 - 16p^{12} - 92p^{16} + 192p^{20} + 112p^{24} - 720p^{28} + 698p^{32} + 384p^{36}$ $-1552p^{40} + 1744p^{44} - 1116p^{48} + 448p^{52} - 112p^{56} + 16p^{60} - p^{64}$
$\boldsymbol{C}^{(1,0,1,1,0,0)}$	$32p^8 - 96p^{12} - 120p^{16} + 1424p^{20} - 4424p^{24} + 8304p^{28} - 10894p^{32} + 10560p^{36}$ $-7744p^{40} + 4320p^{44} - 1816p^{48} + 560p^{52} - 120p^{56} + 16p^{60} - p^{64}$
$\boldsymbol{C}^{(0,1,1,1,0,0)}$	$64p^8 - 448p^{12} + 1680p^{16} - 4256p^{20} + 7952p^{24} - 11424p^{28} + 12868p^{32} - 11440p^{36}$ $+8008p^{40} - 4368p^{44} + 1820p^{48} - 560p^{52} + 120p^{56} - 16p^{60} + p^6 4$
$\boldsymbol{C}^{(1,1,0,0,1,0)}$	$64p^8 - 128p^{10} + 96p^{12} - 32p^{14} - 1532p^{16} + 6144p^{18} - 10752p^{20} + 10752p^{22} + 9664p^{24} - 95616p^{26}$ $+269664p^{28} - 450464p^{30} + 441338p^{32} + 118784p^{34} - 1729536p^{36} + 4486144p^{38} - 7423040p^{40}$ $+8938624p^{42} - 8199136p^{44} + 5857184p^{46} - 3294716p^{48} + 1464320p^{50} -$ $512512p^{52} + 139776p^{54} - 29120p^{56} + 4480p^{58} - 480p^{60} + 32p^{62} - p^{64}$
$\boldsymbol{C}^{(1,0,1,0,1,0)}$	$128p^8 - 256p^{10} - 320p^{12} + 1472p^{14} - 5496p^{16} + 15616p^{18} + 7200p^{20} - 138656p^{22} + 254648p^{24} + 104576p^{26} -$ $1062432p^{28} + 1528032p^{30} - 17422p^{32} - 3037184p^{34} + 4820608p^{36} - 3005056p^{38} - 1494624p^{40}$ $+5473536p^{42} - 6668992p^{44} + 5345344p^{46} - 3166616p^{48} + 1441024p^{50}$ $-509600p^{52} + 139552p^{54} - 29112p^{56} + 4480p^{58} - 480p^{60} + 32p^{62} - p^{64}$
$\boldsymbol{C}^{(0,1,1,0,1,0)}$	$256p^8 - 512p^{10} - 2688p^{12} + 9088p^{14} + 5904p^{16} - 61952p^{18} + 61632p^{20} + 165440p^{22} - 454320p^{24} + 141568p^{26}$ $+1016256p^{28} - 1785920p^{30} + 443716p^{32} + 2654720p^{34} - 4588384p^{36} + 2904160p^{38} + 1526280p^{40}$ $-5480576p^{42} + 6670048p^{44} - 5345440p^{46} + 3166620p^{48} - 1441024p^{50}$ $+509600p^{52} - 139552p^{54} + 29112p^{56} - 4480p^{58} + 480p^{60} - 32p^{62} + p^{64}$
$\boldsymbol{C}^{(1,0,0,1,1,0)}$	$512p^8 - 3072p^{10} + 8960p^{12} - 16640p^{14} - 43744p^{16} + 765312p^{18} - 4637568p^{20} + 18013760p^{22} - 51204560p^{24}$ $+113425312p^{26} - 203255568p^{28} + 301928416p^{30} - 378028286p^{32} + 403556352p^{34} - 370208768p^{36}$ $+293307392p^{38} - 201225472p^{40} + 119608832p^{42} - 61506048p^{44} + 27263232p^{46} - 10354528p^{48}$ $+3339648p^{50} - 903168p^{52} + 201152p^{54} - 35952p^{56} + 4960p^{58} - 496p^{60} + 32p^{62} - p^{64}$
$\boldsymbol{C}^{(0,1,0,1,1,0)}$	$1024p^8 - 6144p^{10} + 1536p^{12} + 114176p^{14} - 542144p^{16} + 1039104p^{18} + 797952p^{20} - 11825024p^{22}$ $+43312992p^{24} - 105270976p^{26} + 196334304p^{28} - 297069632p^{30} + 375202628p^{32} - 402199296p^{34}$ $+369674944p^{36} - 293137856p^{38} + 201182992p^{40} - 119600736p^{42} + 61504944p^{44} - 27263136p^{46}$ $+10354524p^{48} - 3339648p^{50} + 903168p^{52} - 201152p^{54} + 35952p^{56} - 4960p^{58} + 496p^{60} - 32p^{62} + p^{64}$
$\boldsymbol{C}^{(0,0,1,1,1,0)}$	$4096p^8 - 57344p^{10} + 415744p^{12} - 2050048p^{14} + 7653632p^{16} - 22887424p^{18} + 56715264p^{20}$ $-119066112p^{22} + 214987136p^{24} - 337392384p^{26} + 463591296p^{28} - 560492800p^{30} + 598138512p^{32}$ $-564338304p^{34} + 470897216p^{36} - 347203584p^{38} + 225750336p^{40} - 129016384p^{42}$ $+64511136p^{44} - 28048704p^{46} + 10518296p^{48} - 3365856p^{50} + 906192p^{52} - 201376p^{54}$ $+35960p^{56} - 4960p^{58} + 496p^{60} - 32p^{62} + p^{64}$

For hammocks we rely on the results just published in [7] as well on the ones for $\boldsymbol{H}_{8,8}$ and $\boldsymbol{H}^+_{8,8}$ recently reported in [10], [1]. The associated reliability polynomials were computed using our own recursive depth-first traversal of a binary tree algorithm, and are reported in Table 2.

Table 2: Reliability polynomials for $\boldsymbol{H}^+_{8,8}$ and $\boldsymbol{H}_{8,8}$.

$\boldsymbol{N}$	$R(\boldsymbol{N};p)$
$\boldsymbol{H}_{8,8}$	$650p^8 - 580p^9 + 908p^{10} - 6880p^{11} + 4628p^{12} - 12104p^{13} + 31618p^{14} + 372p^{15} + 10594p^{16} + 196688p^{17} - 404536p^{18}$ $+915388p^{19} - 5608084p^{20} + 7645892p^{21} - 12887466p^{22} + 56185408p^{23} - 61734474p^{24} + 83601572p^{25}$ $-412397124p^{26} + 272424760p^{27} + 274694424p^{28} + 1746408000p^{29} - 221980272p^{30} - 12868843904p^{31}$ $+11123958002p^{32} - 11120041788p^{33} + 156260690872p^{34} - 378857360436p^{35} + 264833158482p^{36}$ $-60539595908p^{37} + 345161573768p^{38} + 1581294699620p^{39} - 10357633700988p^{40} + 19594821559752p^{41}$ $-7205288635438p^{42} - 36413539831436p^{43} + 75842387925382p^{44} - 55098726855452p^{45} - 30641343744796p^{46}$ $+111186328020944p^{47} - 111483252211446p^{48} + 33001245825824p^{49} + 53388841078258p^{50} - 85170175686428p^{51}$ $+59759032847258p^{52} - 15870886733412p^{53} - 12944378218252p^{54} + 19685718553176p^{55} - 14268363534224p^{56}$ $+7162471625508p^{57} - 2694331712884p^{58} + 775005119032p^{59} - 169487849178p^{60} + 27440435336p^{61}$ $-3113881376p^{62} + 221751056p^{63} - 7474305p^{64}$
$\boldsymbol{H}^+_{8,8}$	$720p^8 - 720p^9 + 1052p^{10} - 7864p^{11} + 6482p^{12} - 16012p^{13} + 43042p^{14} - 16492p^{15} + 35378p^{16} + 202080p^{17} - 418416p^{18}$ $+840000p^{19} - 6142350p^{20} + 7346188p^{21} - 11370674p^{22} + 74129792p^{23} - 100005860p^{24} + 118520824p^{25}$ $-656753496p^{26} + 1014391664p^{27} - 1060302334p^{28} + 5318496368p^{29} - 8451329352p^{30} + 2451624096p^{31}$ $-37298482094p^{32} + 119852403404p^{33} - 23621628548p^{34} - 197506250928p^{35} - 337635320852p^{36}$ $+1438498163768p^{37} - 67797908976p^{38} - 4198255335740p^{39} + 3015682674902p^{40} + 8881103035456p^{41}$ $-15082194064786p^{42} - 3506508481748p^{43} + 18452358491432p^{44} + 39640921395644p^{45} - 181496618059380p^{46}$ $+286828005143040p^{47} - 191038611524520p^{48} - 138468526731136p^{49} + 542912067034010p^{50} - 803232038481876p^{51}$ $+814719587176720p^{52} - 634769854840740p^{53} + 396340290321940p^{54} - 202017905414696p^{55} + 84634369170678p^{56}$ $-29121695246028p^{57} + 8171460088944p^{58} - 1843848199008p^{59} + 327008804562p^{60} - 43949841128p^{61}$ $+4211763728p^{62} - 256604464p^{63} + 7474305p^{64}$

4.2 Figures-of-merit

To compare compositions with hammocks we will rely on several *FoMs*:

1. The well-known *Reliability Improvement Index* (RII) introduced by Klaschka in [15], [16];
2. The steepness of $R(\boldsymbol{N};p)$ mentioned by Moore and Shannon [20];
3. The variation of $R(\boldsymbol{N};p)$, which was mentioned in [10].

Reliability Improvement Index

The *Reliability Improvement Index* is defined [15], [16] for any network $\boldsymbol{N}$ as

$$\mathrm{RII}(\boldsymbol{N}) = \frac{\log(p)}{\log\left(R(\boldsymbol{N};p)\right)}. \tag{10}$$

The RII is a measure of the reliability increase produced by a network $\boldsymbol{N}$ and was used in [7] to estimate how much matchstick minimal networks improve on a single device.

Steepness of the reliability polynomials

The *ideal* reliability function proposed by Moore and Shannon is the staircase function:

$$\chi(p) = \begin{cases} 0 & 0 \le p \le 0.5 \\ 1 & 0.5 < p \le 1 \end{cases}$$

One of the goal of [20] was to identify networks having reliability polynomials exhibiting steep $0 \to 1$ transitions. We define FoM_1 as:

$$FoM_1 = \max_{p \in [0,1]} R'(\boldsymbol{N}; p). \tag{11}$$

Since the transition point might be important, we give a finer FoM for the steepness of the polynomials. This is an enhancement over FoM_1 which measures how steep is the reliability polynomial as well as how far from 0.5 is the threshold. So, in general FoM_1^* is equal to FoM_1 weighted by a function of the distance between p_0 and 0.5. Here, we choose a very simple function, that is

$$FoM_1^* = \max_{p \in [0,1]} R'(\boldsymbol{N}; p) \cdot \frac{1}{|0.5 - p_0|}, \tag{12}$$

where p_0 is the point where the maximum is achieved. Notice that in our case this is well-defined since no network studied in this paper has $p_0 = 0.5$. But this is no longer the case for self-dual networks where $p_0 = 0.5$, and a modified FoM_1 should be proposed.

Variation of the reliability polynomial

Another FoM is introduced in this paper. It is related to the variation achieved by a reliability polynomial in a given interval. We shall use the area under $R'(\boldsymbol{N}; p)$ in a given symmetric interval (with respect to 0.5). This is exactly the variation of $R(\boldsymbol{N}; p)$ on that interval, hence for any $\boldsymbol{N}$:

$$FoM_2(p_0) = R(\boldsymbol{N}; 1 - p_0) - R(\boldsymbol{N}; p_0) = \int_{p_0}^{1-p_0} R'(\boldsymbol{N}; p) dp. \tag{13}$$

This FoM_2 is well-defined for the staircase function since χ may be written as the integral of the delta Dirac function over the sub-domain $[0, 1]$. Therefore, FoM_2 is the area under the delta Dirac function, between two symmetric points t and $1 - t$, with $0 \leq t < 0.5$.

4.3 Numerical results

Reliability improvement index

The first set of simulations was performed over the whole set $\mathcal{C}_{26}$ as well as for the two hammocks under investigations, $\boldsymbol{H}_{8,8}$ and $\boldsymbol{H}^+_{8,8}$ (see Table 2). Using eq. (10) we have calculated all the RII*s*. These can be seen in Fig. 5,6,7. In Fig. 5 the scale is linear to get a clear picture of the very large RII values for p close to 1. In includes only the square networks, more exactly $\boldsymbol{N} \in \mathcal{C}_{8,8}$ in blue and $\boldsymbol{N} = \boldsymbol{H}_{8,8}, \boldsymbol{H}^+_{8,8}$ in red. A zoom in on the region of interest is shown in Fig. 6, where the yellow horizontal line at RII $= 1$ represents the border between networks that improve reliability and networks which do not. Finally, the complete picture (Fig. 7), in log scale, includes all networks $\boldsymbol{N} \in \mathcal{C}_{26}$, the non-square ones being plotted in orange.

This figure shows a wide range of variation for RII*s*. Among these, those which go below RII $= 1$ are not improving over a single device, which means they should not be used. This is in support of selecting square networks which tend to stick together close to RII $= 1$ when $p = 0.5$.

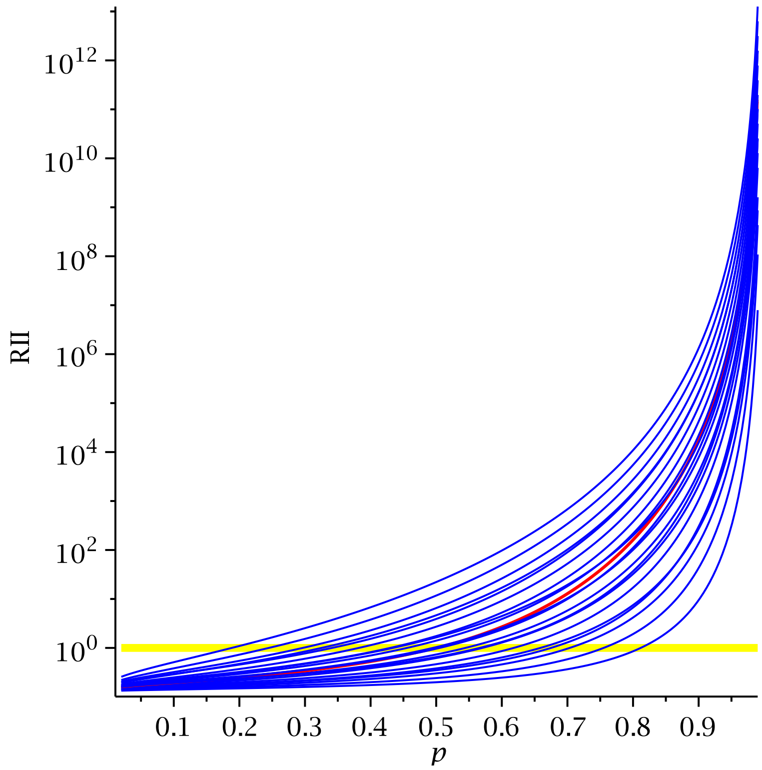


Figure 5: $\mathrm{RII}(\boldsymbol{N})$ for $\boldsymbol{N} \in \mathcal{C}_{8,8}$ (blue) and $\boldsymbol{N} = \boldsymbol{H}_{8,8}, \boldsymbol{H}^{+}_{8,8}$ (red).

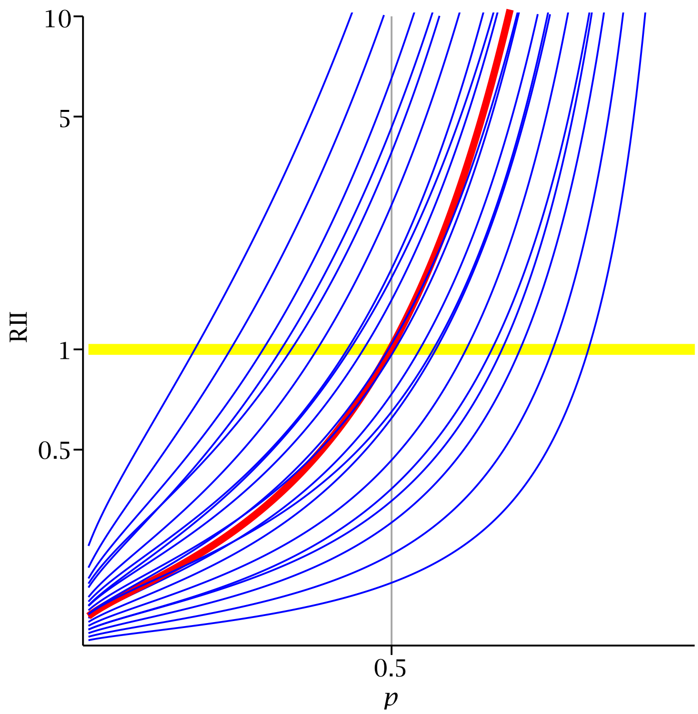


Figure 6: Zoom on $\mathrm{RII}(\boldsymbol{N})$ for $\boldsymbol{N} \in \mathcal{C}_{8,8}$ (blue) and $\boldsymbol{N} = \boldsymbol{H}_{8,8}, \boldsymbol{H}^{+}_{8,8}$ (red).

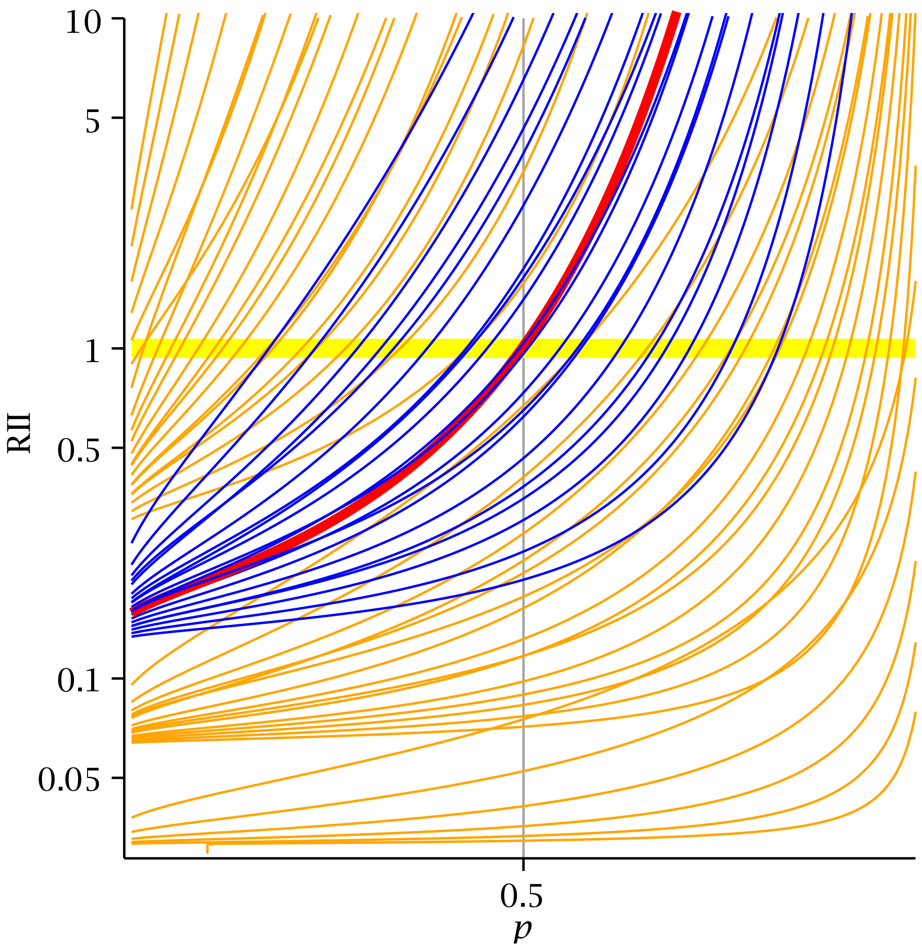


Figure 7: RII($\boldsymbol{N}$) for $\boldsymbol{N} \in \mathcal{C}_{8,8}$ (blue), $\boldsymbol{N} = \boldsymbol{H}_{8,8}, \boldsymbol{H}^{+}_{8,8}$ (red) and $\boldsymbol{N} \in \mathcal{C}_{2^6} \setminus \mathcal{C}_{8,8}$ (orange) in log scale.

Steepness of the reliability polynomials

In Fig. 8 we plot $R'(\boldsymbol{N};p)$ for all $\boldsymbol{N} \in \mathcal{C}_{8,8}$, as well as $\boldsymbol{N} = \boldsymbol{H}_{8,8}$ and $\boldsymbol{N} = \boldsymbol{H}^{+}_{8,8}$. We notice that $\max\limits_{p\in[0,1]} R'(\boldsymbol{H}_{8,8};p) = \max\limits_{p\in[0,1]} R'(\boldsymbol{H}^{+}_{8,8};p) = 3.75252$ and is reached at $p_0 = 0.501745$, respectively $p_0 = 0.498255$. For compositions we have:

$$\max_{\boldsymbol{C}\in\mathcal{C}_{8,8}} \left(\max_{p\in[0,1]} R'(\boldsymbol{C};p) \right) = 4.1035,$$

which is achieved by $\boldsymbol{u} = (1,1,1,0,0,0)$ at $p_0 = 0.760$.

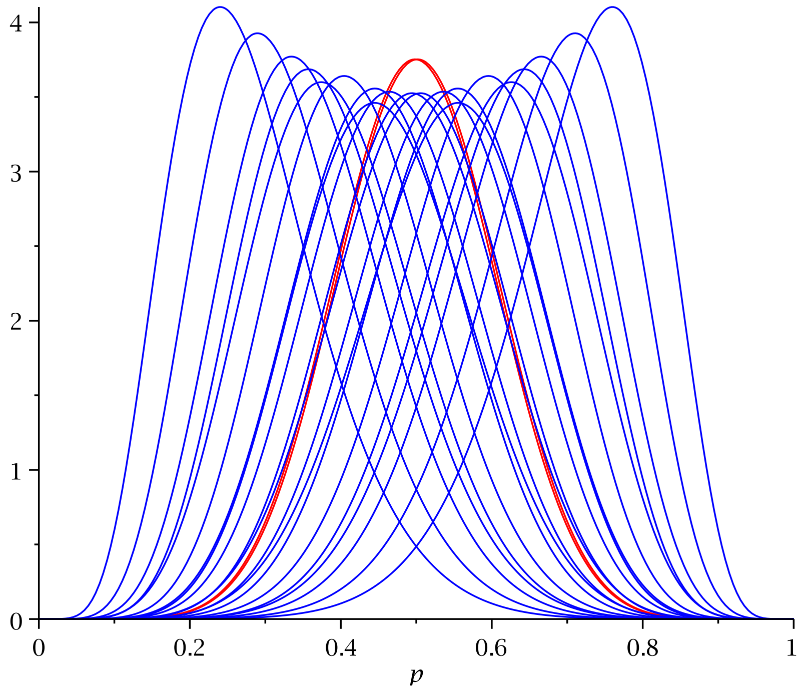


Figure 8: $R'(\boldsymbol{N}; p)$

This result shows the limitation of this FoM, as it does not take into account how far the threshold point is from the desired point, that is 0.5. That is why the enhanced version of this FoM, namely FoM_1^* gives better results in comparing reliability. This fact is illustrated in Fig. 9 and 10.

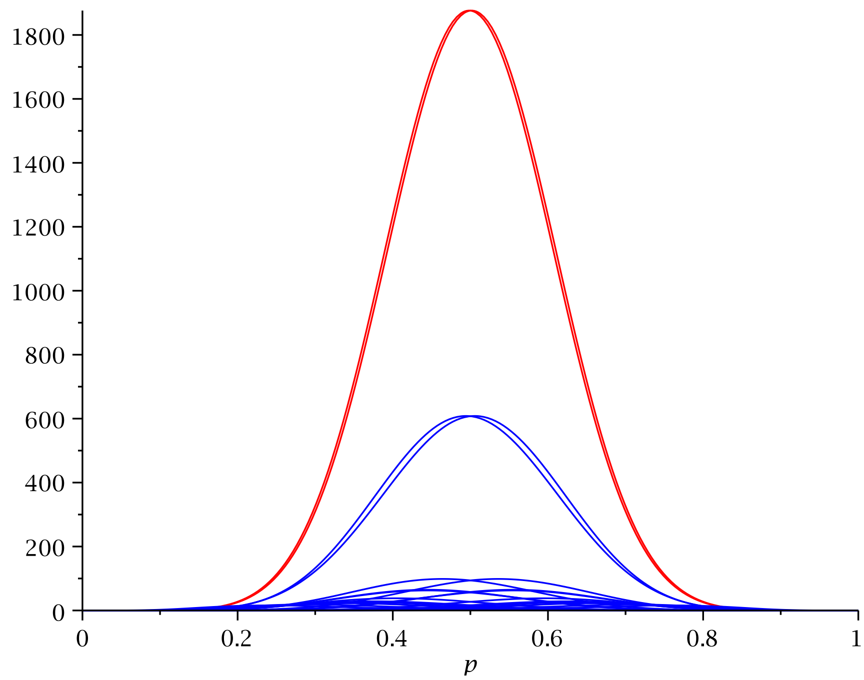


Figure 9: $\dfrac{R'(\boldsymbol{N}; p)}{|p_0 - 0.5|}$

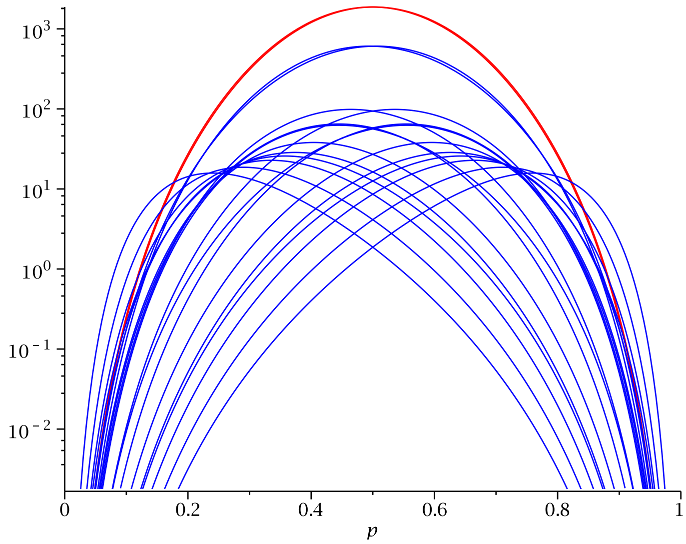


Figure 10: $\log\left(\dfrac{R'(\boldsymbol{N};p)}{|p_0-0.5|}\right)$

We observe from Fig. 9 and 10 that hammocks are "better" than compositions with respect to FoM_1^*. Indeed, for hammocks we obtain $FoM_1^*(\boldsymbol{H}_{8,8}) = 1876.25$ and for compositions we have

$$\max_{\boldsymbol{C}\in\mathcal{C}_{8,8}}\left(\frac{\max\limits_{p\in[0,1]} R'(\boldsymbol{C};p)}{|p_0-0.5|}\right) = 587.56,$$

which is achieved by $\boldsymbol{u} = (0,1,0,1,1,0)$ and $\boldsymbol{u} = (1,0,1,0,0,1)$.

Variation of the reliability polynomials

In Fig. 11 we plot $R(\boldsymbol{N};1-p_0) - R(\boldsymbol{N};p_0)$ as a function of $0 \le p_0 < 0.5$ for $\boldsymbol{H}_{8,8}$ and $\boldsymbol{H}_{8,8}^+$, and $\boldsymbol{C}\in\mathcal{C}_{8,8}$.

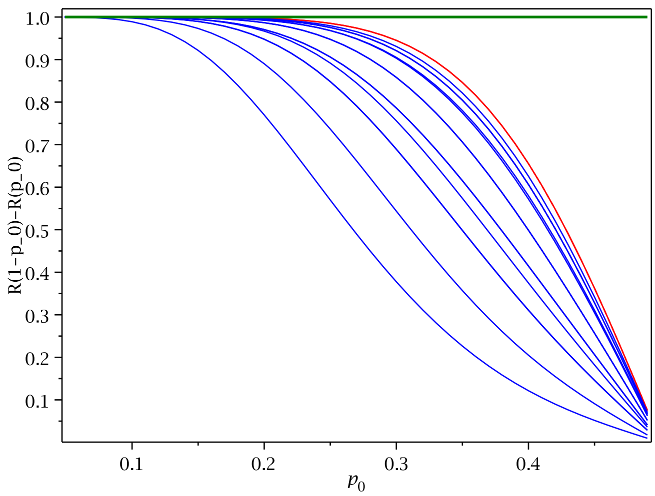


Figure 11: $R(\boldsymbol{N};1-p_0) - R(\boldsymbol{N};p_0)$ as a function of p_0 $(0 \le p_0 < 0.5)$;$\boldsymbol{N} = \boldsymbol{H}_{8,8}$ and $\boldsymbol{H}_{8,8}^+$ (red), $\boldsymbol{N} = \boldsymbol{C}\in\mathcal{C}_{8,8}$ (blue), and $\chi(1-p_0) - \chi(p_0)$ (green).

We notice a difference between the curves starting to develop from $p_0 = 0.25$ onwards. In Table 3 we report the exact values for $p_0 = 0.25$, which correspond to $R(0.75) - R(0.25)$. The two hammocks we have considered here achieve the same value $R(\boldsymbol{H}_{8,8}; 0.75) - R(\boldsymbol{H}_{8,8}; 0.25) = 0.985173$. For compositions the best value

$$\max_{\boldsymbol{u} \in \{0,1\}^6} R(\boldsymbol{C}^{\boldsymbol{u}}; 0.75) - R(\boldsymbol{C}^{\boldsymbol{u}}; 0.25) = 0.979507,$$

is achieved for $\boldsymbol{u} = (0,1,0,1,1,0)$ and $\boldsymbol{u} = (1,0,1,0,0,1)$. It should be mentioned that the same two compositions achieve the best values for FoM_1^*. In fact, $FoM_2(0.25)$ correlates perfectly with FoM_1^*. Indeed, if we totally order compositions and hammocks with respect to FoM_1^*, then the same order holds for $FoM_2(0.25)$. And as expected, the two FoMs point out to the same network as being the most reliable, namely the hammock.

Recall that one of the leading arguments when comparing networks was the restriction of the number of devices, which in our case study is always $n = 64$. Now, if we plot $FoM_2(0.25)$ as a function of the number of wires ω, we see that one of the best compositions, namely $\boldsymbol{u} = (0,1,0,1,1,0)$ has fewer wires than the hammocks (see Fig. 12).

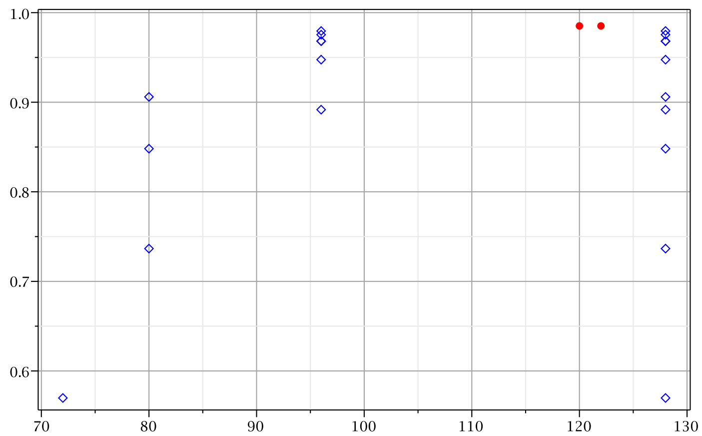


Figure 12: $R(\boldsymbol{N}; 0.75) - R(\boldsymbol{N}; 0.25)$ as a function of the number of wires ω. $\boldsymbol{N} = \boldsymbol{H}_{\mathbf{8,8}}$ and $\boldsymbol{N} = \boldsymbol{H}^{+}_{\mathbf{8,8}}$ (red), and $\boldsymbol{N} = \boldsymbol{C}$ with $\boldsymbol{C} \in \mathcal{C}_{8,8}$ (blue)

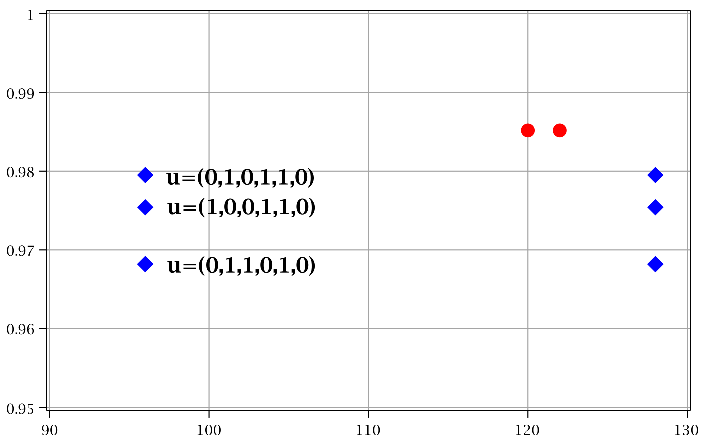


Figure 13: Zoom on the values of $R(\boldsymbol{N}; 0.75) - R(\boldsymbol{N}; 0.25)$ in the vicinity of the maximum values

Table 3: Figure-of-merit for $\boldsymbol{C} \in \mathcal{C}_{8,8}$, $\boldsymbol{H}_{8,8}$, and $\boldsymbol{H}^{+}_{8,8}$.

ω	$\boldsymbol{N}$	p_0	$\max_{p\in[0,1]} R'(\boldsymbol{N};p)$	$\frac{\max_{p\in[0,1]} R'(\boldsymbol{N};p)}{\lvert p_0 - 0.5\rvert}$	$R(\boldsymbol{N};0.75) - R(\boldsymbol{N};0.25)$
120	$\boldsymbol{H}_{8,8}$	0.498	3.7525	1876.25	0.985173
122	$\boldsymbol{H}^{+}_{8,8}$	0.502	3.7525	1876.25	0.985173
72	$\boldsymbol{C}^{(1,1,1,0,0,0)}$	0.760	4.1035	15.78	0.569843
80	$\boldsymbol{C}^{(1,1,0,1,0,0)}$	0.710	3.9273	18.70	0.736601
	$\boldsymbol{C}^{(1,0,1,1,0,0)}$	0.665	3.7709	22.85	0.848154
	$\boldsymbol{C}^{(0,1,1,1,0,0)}$	0.626	3.5995	28.56	0.905953
96	$\boldsymbol{C}^{(1,1,0,0,1,0)}$	0.643	3.6861	25.77	0.891705
	$\boldsymbol{C}^{(1,0,1,0,1,0)}$	0.596	3.6409	37.92	0.947539
	$\boldsymbol{C}^{(0,1,1,0,1,0)}$	0.555	3.5568	64.66	0.968192
	$\boldsymbol{C}^{(1,0,0,1,1,0)}$	0.536	3.5354	98.20	0.975413
	$\boldsymbol{C}^{(0,1,0,1,1,0)}$	0.494	3.5254	587.56	0.979507
	$\boldsymbol{C}^{(0,0,1,1,1,0)}$	0.445	3.4606	62.92	0.968192
128	$\boldsymbol{C}^{(1,1,0,0,0,1)}$	0.555	3.4606	62.92	0.968192
	$\boldsymbol{C}^{(1,0,1,0,0,1)}$	0.506	3.5254	587.56	0.979507
	$\boldsymbol{C}^{(0,1,1,0,0,1)}$	0.464	3.5354	98.20	0.975413
	$\boldsymbol{C}^{(1,0,0,1,0,1)}$	0.445	3.5568	64.66	0.968192
	$\boldsymbol{C}^{(0,1,0,1,0,1)}$	0.404	3.6409	37.92	0.947538
	$\boldsymbol{C}^{(0,0,1,1,0,1)}$	0.357	3.6861	25.77	0.891705
	$\boldsymbol{C}^{(1,0,0,0,1,1)}$	0.374	3.5995	28.56	0.905953
	$\boldsymbol{C}^{(0,1,0,0,1,1)}$	0.335	3.7709	22.85	0.848154
	$\boldsymbol{C}^{(0,0,1,0,1,1)}$	0.290	3.9273	18.70	0.736601
	$\boldsymbol{C}^{(0,0,0,1,1,1)}$	0.240	4.1035	15.78	0.569843

Strong points of compositions

Notice that for $\boldsymbol{u} = (0,1,0,1,1,0)$ the corresponding composition comes really close to $\boldsymbol{H}_{8,8}$ and has an advantage over $\boldsymbol{H}_{8,8}$, in that $\boldsymbol{C}^{(0,1,0,1,1,0)}$ has only 96 wires, while $\boldsymbol{H}_{8,8}$ has 120. From a computational point of view, $R(\boldsymbol{C}^{\boldsymbol{u}};p)$ has several advantages over $R(\boldsymbol{H}_{8,8};p)$.

- Firstly, notice that the order of magnitude of the largest coefficient is 10^9 for $\boldsymbol{C}^{\boldsymbol{u}}$ compared with 10^{14} for $\boldsymbol{H}_{8,8}$.

- Secondly, the reliability polynomials for compositions are sparser than $R(\boldsymbol{H}_{8,8};p)$. This is due to the fact that $R(\boldsymbol{C}^{\boldsymbol{u}};p)$ have non-zero coefficients only for even powers of p, i.e., 29 non-zero coefficients versus 57 for $R(\boldsymbol{H}_{8,8};p)$.

- Thirdly the absolute values of the coefficients of $R(\boldsymbol{H}_{8,8};p)$ are larger on average than the coefficients of $R(\boldsymbol{C}^{\boldsymbol{u}};p)$. For the case $m = 6$, the average value of a coefficient is of the order 1.3×10^{13} for hammocks, compared with 8.6×10^{7} for compositions.

From the computational point of view all these arguments favor compositions over hammocks.

5 Conclusions

In this article we have proposed and analyzed two-terminal networks generated through the repeated composition of the simplest series and parallel networks. We have detailed several structural properties of such networks and have presented an efficient method for computing their associated reliability polynomials.

Compositions were compared with hammocks according to three *FoM*s: RII, the slope of the reliability polynomials and their variations. For the particular cases considered here we have observed that compositions come very close to hammocks, without surpassing them. Still, compositions of series and parallel present several advantages. There are compositions performing almost as well as hammocks, while having fewer wires than hammocks, for the same number of devices. We also noticed that there is a significant computational gap, the reliability polynomials of compositions being much simpler/easier to compute and analyze exactly.

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